Announcements

1) New Webwork up tomorrow

Recall L'Hopital's Rule

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

with appropriate hypotheses.

$$\left(\frac{0}{0}, \frac{\pm\infty}{\pm\infty} \right)$$

Example 1. $\lim_{x \to 28} \dfrac{\sqrt[3]{x-1} - 3}{x - 28}$

$$\lim_{x \to 28} \left(\sqrt[3]{x-1} - 3\right) = 0$$

$$\lim_{x \to 28} (x - 28) = 0$$

l'Hopital's rule?

$$\lim_{x \to 28} \frac{\sqrt[3]{x-1} - 3}{x - 28}$$

$$\overset{L'H}{=} \lim_{x \to 28} \frac{\frac{1}{3}(x-1)^{-2/3}}{1}$$

$$= \frac{1}{3}\, 27^{-2/3} = \frac{1}{27}$$

Why is this illegitimate?

By definition,

if $f(x) = \sqrt[3]{x-1}$,

$$f'(28) = \lim_{x \to \infty} \frac{\sqrt[3]{x-1} - 3}{x-28}$$

Let $g(x) = x - 28$.

$$\lim_{x \to 28} \frac{\sqrt[3]{x-1} - 3}{x-28} = \lim_{x \to 28} \frac{f'(x)}{g'(x)}$$

$$= \lim_{x \to 28} f'(x),$$

Since f' is continuous at $x=28$,

$$\lim_{x\to 28} f'(x) = f'(28)$$

$$= \lim_{x\to 28} \frac{\sqrt[3]{x-1}-3}{x-28}$$

back to where you started!

Example 2: $\lim_{x \to 0} \frac{\sin x - x}{4x^3}$

$\lim_{x \to 0} (\sin x - x) = 0$

$\lim_{x \to 0} 4x^3 = 0$

Use l'Hopital's Rule.

$$\lim_{x\to 0} \frac{\sin x - x}{4x^3}$$

$$\overset{l'H}{=} \lim_{x\to 0} \frac{\cos(x) - 1}{12x^2} = \frac{0}{0}$$

Use l'Hopital's rule again

$$\overset{l'H}{=} \lim_{x\to 0} \frac{-\sin(x)}{24x} = \frac{0}{0}$$

One more time

$$\overset{l'H}{=} \lim_{x\to 0} \frac{-\cos(x)}{24} = \boxed{-\frac{1}{24}}$$

L'Hopital's Rule can also be used on "indeterminate products" of the form

$$\lim_{x \to a} \left(f(x) g(x) \right)$$

where $\lim_{x \to a} f(x) = 0$, $\lim_{x \to a} g(x) = \pm\infty$.

Trick: make a quotient by flipping one function to the denominator. Use l'Hopital.

Warning: You have a choice of which function to flip. There is usually a good choice and a bad choice!

Example 3: $\lim_{x \to \frac{\pi}{2}^+} \sec(x)(2x - \pi)$

$$\lim_{x \to \frac{\pi}{2}^+} (2x - \pi) = 0$$

$$\lim_{x \to \frac{\pi}{2}^+} \sec(x) = \lim_{x \to \frac{\pi}{2}^+} \frac{1}{\cos(x)}$$

$$= -\infty$$

Rewrite as a quotient, use l'Hopital's Rule.

Try flipping sec(x).

$$\lim_{x \to \frac{\pi}{2}^+} \sec(x)(2x - \pi)$$

$$= \lim_{x \to \frac{\pi}{2}^+} \frac{2x - \pi}{\cos(x)} = \frac{0}{0}$$

$$\stackrel{\text{L'H}}{=} \lim_{x \to \frac{\pi}{2}^+} \frac{2}{-\sin(x)} = \boxed{-2}$$

What if you tried the other way?

$$\lim_{x \to \frac{\pi}{2}^+} \sec(x)(2x-\pi)$$

$$= \lim_{x \to \frac{\pi}{2}^+} \frac{\sec(x)}{(2x-\pi)^{-1}} = \frac{-\infty}{\infty}$$

$$\stackrel{L'H}{=} \lim_{x \to \frac{\pi}{2}^+} \frac{\sec(x)\tan(x)}{-1 \cdot (2x-\pi)^{-2} \cdot 2}$$

Harder than what you started with! Quit.

Example 4: L'Hopital's rule

sometimes won't help!

$$\lim_{x\to 0^+} \frac{\sqrt{x^2-x^3}}{x}$$

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{1}{2}(x^2-x^3)^{-1/2}(2x-3x^2)$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{2}(2x-3x^2)}{\sqrt{x^2-x^3}} = \frac{0}{0}$$

$$\stackrel{L'H}{=} \lim_{x\to 0^+} \frac{\cancel{1}}{\cancel{2}} \frac{(2-6x)}{\cancel{1/2}(x^2-x^3)^{-1/2}(2x-3x^2)}$$

$$= \lim_{x\to 0^+} \frac{(2-6x)\sqrt{x^2-x^3}}{2x-3x^2} = \frac{0}{0}$$

In fact, $\lim_{x\to 0^+} \frac{\sqrt{x^2-x^3}}{x}$

$$= \lim_{x\to 0^+} \frac{\sqrt{x^2(1-x)}}{x}$$

$$= \lim_{x\to 0^+} \frac{\cancel{x}\sqrt{1-x}}{\cancel{x}}$$

$$= \boxed{1}$$

Summary

1) l'Hopital's Rule is for quotients of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

2) You can also use it on products of the form $\pm\infty \cdot 0$ - be careful when making a quotient, you **MUST** make a quotient to use the rule.

3) Watch out for square roots (last example)

The Mean Value Theorem

Recall: "mean" = "average"

(Section 3.2)

Idea Behind the Theorem

You travel from Dearborn to chicago in 4 ½ hours. This is 57.7 (approximately) miles per hour average velocity.

The mean value theorem says that at some point on your trip, you had to have an instantaneous velocity of 57.7 mph.

Like the Intermediate Value Theorem, you don't know when exactly you hit that speed.

Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a point c with $a < c < b$ and

$$\underbrace{\boxed{f'(c)}}_{\text{Instantaneous}} \boxed{= \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{average}}}$$

Rolle's Theorem: (simplification of Mean Value Theorem)

If f is continuous on $[a, b]$

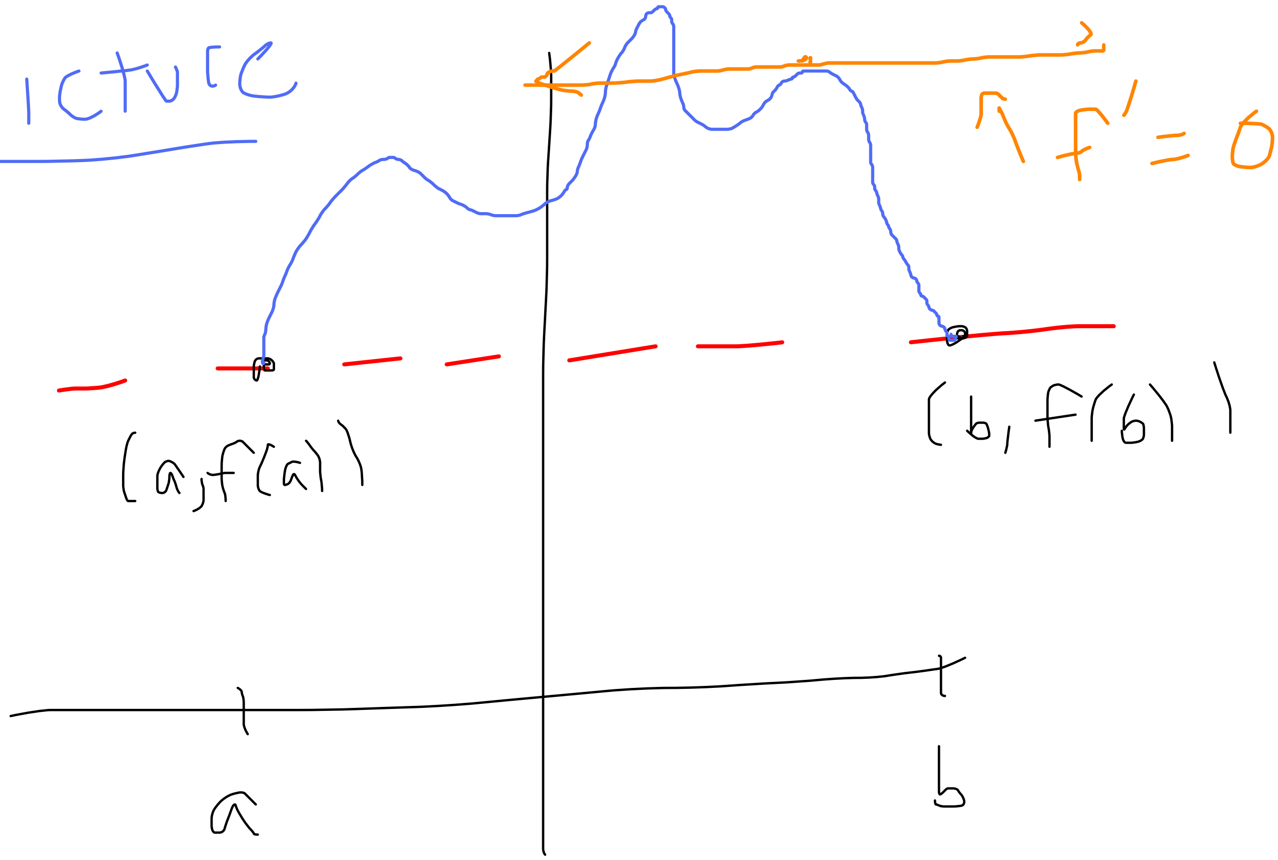
and differentiable on (a, b)

and $f(a) = f(b)$, then there

is a point c with $a < c < b$

and $\boxed{f'(c) = 0}$

Picture



Why is Rolle's Theorem true?

3 cases

1) f is constant on $[a,b]$

Then $f'(c)=0$ for all

c in (a,b)

2) f is nonconstant, max

We assume f is non-constant.

If f has a local maximum in (a,b), then by Fermat's Theorem, since f' exists on (a,b), f' must be zero at that point.

3) f is nonconstant, min

Same argument as 2)

Getting the Mean Value Theorem from Rolle's Theorem

Let $g(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a}\right)x$

You can check that

$$g(a) = g(b).$$

By Rolle's Theorem, there is a point c with

$$a < c < b \text{ and } g'(c) = 0.$$

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

so $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$